

Neutrino dispersion in external magnetic fields

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We calculate the neutrino self-energy operator $\Sigma(p)$ in the presence of a magnetic field B . In particular, we consider the weak-field limit $eB \ll m_\ell^2$, where m_ℓ is the charged-lepton mass corresponding to the neutrino flavor ν_ℓ , and we consider a “moderate field” $m_\ell^2 \ll eB \ll m_W^2$. Our results differ substantially from the previous literature. For a moderate field, we show that it is crucial to include the contributions from all Landau levels of the intermediate charged lepton, not just the ground state. For the conditions of the early universe where the background medium consists of a charge-symmetric plasma, the pure B -field contribution to the neutrino dispersion relation is proportional to $(eB)^2$ and thus comparable to the contribution of the magnetized plasma.

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I. INTRODUCTION

The presence of matter or electromagnetic fields modifies the dispersion relation of neutrinos in rather subtle ways because these elusive particles interact only by the weak force. However, Wolfenstein was the first to recognize that the feeble matter effect is enough to affect neutrino flavor oscillations in dramatic ways because the neutrino mass differences are very small [1], with practical applications in physics and astrophysics whenever neutrino oscillations are important [2]. The presence of external fields will lead to additional modifications of the neutrino dispersion relation. There is a natural scale for the field strength that is required to have a significant impact on quantum processes, i.e. the critical value

$$B_e = m_e^2/e \approx 4.41 \times 10^{13} \text{ G}. \quad (1)$$

Note that we use natural units where $\hbar = c = 1$ and the Lorentz-Heaviside convention where $\alpha = e^2/4\pi \approx 1/137$ so that $e \approx 0.30 > 0$ is the elementary charge, taken to be positive.

There are reasons to expect that fields of such or even

larger magnitudes can arise in cataclysmic astrophysical events such as supernova explosions or coalescing neutron stars, situations where a gigantic neutrino outflow should also be expected. There are two classes of stars, i.e. soft gamma-ray repeaters (SGR) [3, 4] and anomalous x-ray pulsars (AXP) [5, 6] that are believed to be remnants of such cataclysms and to be magnetars [7], neutron stars with magnetic fields 10^{14} – 10^{15} G. The possible existence of even larger fields of order 10^{16} – 10^{17} G is subject to debate [8–12]. The early universe between the QCD phase transition ($\sim 10^{-5}$ s) and the nucleosynthesis epoch ($\sim 10^{-2}$ – 10^{+2} s) is believed to be yet another natural environment where strong magnetic fields and large neutrino densities could exist simultaneously [13].

The modification of the neutrino dispersion relation in a magnetized astrophysical plasma was studied in the previous literature [14–17]. In particular, a charge-symmetric plasma with $m_e \ll T \ll m_W$ and $B \lesssim T^2$ was considered for the early-universe epoch between the QCD phase transition and big-bang nucleosynthesis. Ignoring the neutrino mass, the dispersion relation for the electron flavor was found to be [16, 17]

$$\frac{E}{|\mathbf{p}|} = 1 + \frac{\sqrt{2} G_F}{3} \left[-\frac{7\pi^2 T^4}{15} \left(\frac{1}{m_Z^2} + \frac{2}{m_W^2} \right) + \frac{T^2 e B}{m_W^2} \cos \phi + \frac{(eB)^2}{2\pi^2 m_W^2} \ln \left(\frac{T^2}{m_e^2} \right) \sin^2 \phi \right], \quad (2)$$

where \mathbf{p} is the neutrino momentum and ϕ is the angle between \mathbf{B} and \mathbf{p} . The first term in Eq. (2) is the dominating pure plasma contribution [18], whereas the second term is caused by the common influence of the plasma and magnetic field [16]. The third term is of second order in $(eB/T^2) \ll 1$ but was included because of the large logarithmic factor $\ln(T/m_e) \gg 1$ [17]. The dispersion relation of Eq. (2) applies to both ν_e and $\bar{\nu}_e$ without sign

change in any of the terms [19].

The B -field induced pure vacuum modification of the neutrino dispersion relation was assumed to be negligible in these papers. However, recently this contribution was calculated for the same early-universe conditions as described above [20, 21]. The dispersion relation obtained

in these papers for both ν_e and $\bar{\nu}_e$ can be expressed as

$$\frac{E}{|\mathbf{p}|} = 1 + \sqrt{2} G_F \frac{eB}{8\pi^2} \sin^2 \phi e^{-p_\perp^2/(2eB)}, \quad (3)$$

where p_\perp is the momentum component perpendicular to the B -field. It is easy to check that this would be the dominant B -field induced contribution by far and thus would lead to important consequences for neutrino physics in media [22–24]. The importance of the question whether the B -field contribution to the neutrino dispersion relation is dominant or negligible calls for an independent calculation.

A literature search reveals that calculations of the neutrino dispersion relation in external B -fields have a long history [25–27]. To compare the different results we introduce the neutrino self-energy operator $\Sigma(p)$ that is defined in terms of the invariant amplitude for the transition $\nu \rightarrow \nu$ by the relation

$$\mathcal{M}(\nu \rightarrow \nu) = -\bar{\nu}(p) \Sigma(p) \nu(p), \quad (4)$$

where p is the neutrino four-momentum. Note that we use the signature $(+, -, -, -)$ for the four-metric. Within the standard model, the general Lorentz structure of $\Sigma(p)$ in the presence of a magnetic field can be expressed in terms of four numerical coefficients a , b , c , and d as

$$\Sigma(p) = [a(p\gamma) + b(p\gamma)_\parallel + c(p\tilde{\varphi}\gamma) + id(p\varphi\gamma)] L, \quad (5)$$

where γ_α are the Dirac matrices in the standard representation and $L = \frac{1}{2}(1 - \gamma_5)$ is the left-handed projection operator. The Lorentz indices of four-vectors and tensors within parentheses are contracted consecutively. For example, $(p\varphi\gamma) = p^\alpha \varphi_{\alpha\beta} \gamma^\beta$. Further, φ is the dimensionless tensor of the electromagnetic field, normalized to the external B -field, whereas $\tilde{\varphi}$ is its dual,

$$\begin{aligned} \varphi_{\alpha\beta} &= \frac{F_{\alpha\beta}}{B}, \\ \tilde{\varphi}_{\alpha\beta} &= \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} \varphi^{\mu\nu}. \end{aligned} \quad (6)$$

Finally, in the frame where only an external magnetic field \mathbf{B} is present, we take the spatial 3-axis to be directed along \mathbf{B} . Four-vectors with the indices \perp and \parallel belong to the Euclidean $\{1, 2\}$ -subspace and the Minkowski $\{0, 3\}$ -subspace, correspondingly. For example, $p_\perp = (0, p_1, p_2, 0)$ and $p_\parallel = (p_0, 0, 0, p_3)$. For any four-vectors X and Y we use the notation

$$\begin{aligned} (XY)_\parallel &= (X \tilde{\varphi} \tilde{\varphi} Y) = X_0 Y_0 - X_3 Y_3, \\ (XY)_\perp &= (X \varphi \varphi Y) = X_1 Y_1 + X_2 Y_2, \\ (XY) &= (XY)_\parallel - (XY)_\perp. \end{aligned} \quad (7)$$

Perturbatively, the matrix element of Eq. (4) corresponds to the Feynman diagram shown in Fig. 1 where double lines denote exact propagators in the external B field. Put another way, the self-energy operator corresponds to this Feynman graph with the external neutrino

lines truncated. The motivation for our work is that the results obtained by different authors at one-loop level do not agree with each other. We anticipate our results in Table I where we show b , c and d obtained by previous authors and from our calculation detailed below.

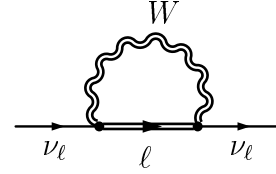


FIG. 1: Feynman diagram for the field-induced contribution to the neutrino self-energy operator. Double lines denote exact propagators of the charged lepton and the W -boson in an external B field. The contribution of the unphysical Higgs particles can be neglected in the limit $m_\ell^2 \ll m_W^2$.

Turning to the interpretation of the coefficients in Eq. (5) we note that a does not have an independent meaning because for small neutrino energies, $E \ll m_W$, when $a(p) = \text{const.}$, the first term in Eq. (5) is fully absorbed by the neutrino wave-function renormalization. The coefficient d corresponds to an induced electric dipole moment and as such can be non-zero only in the presence of the CP-odd field invariant $(F\tilde{F}) = 4\mathbf{E} \cdot \mathbf{B}$. Even in this case it is strongly suppressed [26].

Therefore, in our case of a pure external B -field only the coefficients b and c are relevant for neutrino dispersion. For a massless neutrino the Dirac equation in momentum space is

$$[(p\gamma) - \Sigma(p)] \nu(p) = 0. \quad (8)$$

The dispersion relation follows from

$$\det [(p\gamma) - b(p\gamma)_\parallel L - c(p\tilde{\varphi}\gamma)L] = 0. \quad (9)$$

For both ν and $\bar{\nu}$ this implies the same dispersion relation

$$\frac{E}{|\mathbf{p}|} = 1 + \left(b + \frac{c^2}{2}\right) \sin^2 \phi. \quad (10)$$

Actually, in a perturbative sense the quantity c^2 is of higher order, taking us to the two-loop level. Therefore, to lowest order the dispersion relation of massless neutrinos depends only on b .

For massive neutrinos the situation is more complicated because a transverse B -field induces transitions between positive- and negative-helicity states by the usual spin precession. Put another way, in the standard model neutrinos with mass inevitably have magnetic dipole moments. In this situation it is not particularly illuminating to express the effect of the B -field in terms of a modified dispersion relation because an entirely new phenomenon appears, the mixing of positive- with negative-helicity states. Formally one can still proceed as above by $\Sigma(p) \rightarrow m + \Sigma(p)$ in the Dirac equation and obtain the new dispersion relation. However, the energy eigenstates

TABLE I: Coefficients in Eq. (5) for the neutrino self-energy operator $\Sigma(p)$ in an external B -field.

Authors	Ref.	Field strength ^a	$b \times \frac{\sqrt{2}\pi^2}{G_F}$	$c \times \frac{\sqrt{2}\pi^2}{G_F}$	$d \times \frac{\sqrt{2}\pi^2}{G_F}$
McKeon (1981)	[25]	—	0	$+3eB$	$+2eB$
Erdas & Feldman (1990)	[27]	Moderate	$-\frac{(eB)^2}{3m_W^2} \left(\ln \frac{m_W^2}{m_\ell^2} + \frac{3}{4} \right)$	0	0
Elizalde et al. (2002) ^b	[20]	Moderate	$+\frac{eB}{2}$	$-\frac{eB}{2}$	0
Elizalde et al. (2004) ^b	[21]	Moderate	$+\frac{eB}{4} e^{-p_\perp^2/(2eB)}$	$-\frac{eB}{4} e^{-p_\perp^2/(2eB)}$	0
Our result (2005) ^c		Weak	$-\frac{(eB)^2}{3m_W^2} \left(\ln \frac{m_W^2}{m_\ell^2} + \frac{3}{4} \right)$	$+\frac{3eB}{4}$	0
Our result (2005) ^c		Moderate	$-\frac{(eB)^2}{3m_W^2} \left(\ln \frac{m_W^2}{eB} + 2.542 \right)$	$+\frac{3eB}{4}$	0

^a B is “weak” for $eB \ll m_\ell^2$ and “moderate” for $m_\ell^2 \ll eB \ll m_W^2$.

^bNeutrino momentum range $0 < p_\perp^2 \ll eB$.

^cNeutrino momentum range $0 < p_\perp^2 \ll m_W^2$.

are no longer the left- and right-handed helicity states but rather a superposition that depends on the magnetic field orientation relative to \mathbf{p} .

Therefore, for massive neutrinos the effect of B is better illustrated in terms of the equation of motion of a free neutrino state in an external homogeneous B -field. In this situation neutrino and anti-neutrino states are not connected. The Dirac equation implies for the helicity

amplitudes ν_\pm of a massive neutrino

$$i \frac{\partial}{\partial t} \begin{pmatrix} \nu_+ \\ \nu_- \end{pmatrix} = E_{\mathbf{p}} (1 + \Omega_{\mathbf{p}}^B) \begin{pmatrix} \nu_+ \\ \nu_- \end{pmatrix}, \quad (11)$$

where $E_{\mathbf{p}} = (\mathbf{p}^2 + m^2)^{1/2}$. The dimensionless B -field induced mixing matrix is

$$\begin{aligned} \Omega_{\mathbf{p}}^B &= \frac{1}{2} b \left[\sin^2 \phi \begin{pmatrix} 1-v & 0 \\ 0 & 1+v \end{pmatrix} - \cos \phi \sin \phi \frac{v m}{E_{\mathbf{p}}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \cos^2 \phi \frac{m^2}{E_{\mathbf{p}}^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \\ &+ \frac{m c}{2 E_{\mathbf{p}}} \left[\sin \phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \cos \phi \frac{m}{E_{\mathbf{p}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right], \end{aligned} \quad (12)$$

where $v = |\mathbf{p}|/E$ is the neutrino velocity. The first term would represent at $v = 1$ the energy shift for massless neutrinos of Eq. (10) where we have dropped the higher-order c^2 term. Of course, positive-helicity (right-handed) massless neutrinos do not suffer an energy shift.

The second line is identical to the effect caused by a neutrino magnetic moment

$$\mu_\nu = \frac{m c}{2B}. \quad (13)$$

Therefore, within the standard model the coefficient c is implied by the well-known result for the neutrino mag-

netic moment [28, 29]

$$\mu_\nu = \frac{3e G_F m}{8\pi^2 \sqrt{2}}. \quad (14)$$

Likewise, c can be extracted from Refs. [30, 31], where the neutrino transitions $\nu_i \leftrightarrow \nu_j$ in an external electromagnetic field were investigated. From Eq. (12) and Table I one concludes that for $m \gtrsim 10^{-4} \text{ eV} \times (p_\perp/1 \text{ MeV})(B/B_e)$ the neutrino energy shift becomes essentially non-diagonal.

If the neutrino mass is of Majorana type, the four neutrino components in the full Dirac equation are not independent and the mass-implied term vanishes—Majorana neutrinos do not have a magnetic moment. On the other

hand, once neutrino masses are included one can not avoid flavor mixing so that the situation becomes yet more complicated by the presence of ordinary flavor oscillations and flavor off-diagonal magnetic spin precessions caused by magnetic transition moments which exist for both Dirac and Majorana neutrinos. We will not entangle our discussion with these complications because in the standard model the mass-induced effects all happen on the external neutrino legs in the Feynman graph of Fig. 1. The self-energy operator $\Sigma(p)$ itself is independent of neutrino masses, at least to one-loop order.

We begin in Secs. II and III with the technique to calculate the neutrino self-energy operator by using the charged-lepton and W -boson propagators in a magnetic field. In Sec. IV we calculate the neutrino self-energy contribution from the n -th Landau level of the charged-lepton propagator in combination with the exact W propagator. In contrast to previous assumptions [20, 21] we find that it is not enough to use only the lowest Landau level. In Sec. V we derive explicit results for the neutrino self-energy operator in the limiting cases of a “weak field” $eB \ll m_\ell^2$ and a “moderate field” $m_\ell^2 \ll eB \ll m_W^2$ before concluding in Sec. VI.

II. DEFINITION OF $\Sigma(p)$

The \mathcal{S} matrix element for the transition $\nu \rightarrow \nu$ corresponds to the Feynman diagram shown in Fig. 1 where double lines denote exact propagators in the presence of an external magnetic field. A detailed description of the calculational techniques for quantum processes in external electromagnetic fields can be found in Ref. [32].

For the charged lepton ℓ we consider a negative electric charge $Q_\ell = -e < 0$. The propagator in the presence of a constant and uniform magnetic field is translationally and gauge non-invariant [33]. It can be expressed as

$$S^F(x, y) = e^{i\Phi(x, y)} S(x - y). \quad (15)$$

Here, $S(x - y)$ is the translationally and gauge invariant part of the propagator. The translationally and gauge non-invariant phase can be defined in terms of an integral along an arbitrary contour as

$$\Phi(x, y) = -e \int_x^y d\xi_\mu K^\mu(\xi), \quad (16)$$

where $K^\mu(\xi) = A^\mu(\xi) + \frac{1}{2}F^{\mu\nu}(\xi - y)_\nu$.

The corresponding W -boson propagator can be represented in a similar form

$$G_{\rho\sigma}^F(x, y) = e^{i\Phi(x, y)} G_{\rho\sigma}(x - y), \quad (17)$$

where $\Phi(x, y)$ is also given by Eq. (16).

It is useful to consider the Fourier transforms of the translationally invariant parts of the propagators

$$S(X) = \int \frac{d^4q}{(2\pi)^4} S(q) e^{-iqX}, \quad (18)$$

$$G_{\rho\sigma}(X) = \int \frac{d^4q}{(2\pi)^4} G_{\rho\sigma}(q) e^{-iqX}. \quad (19)$$

The sum of the translationally non-invariant phases of the lepton and W propagators Eqs. (15) and (17) in the loop vanishes,

$$\Phi(x, y) + \Phi(y, x) = 0. \quad (20)$$

This allows one to extract the amplitude \mathcal{M} from the \mathcal{S} matrix element by the standard method,

$$\begin{aligned} \mathcal{M}(\nu \rightarrow \nu) &= \frac{ig^2}{2} \bar{\nu}(p) \gamma^\alpha L \int \frac{d^4q}{(2\pi)^4} S(q) G_{\beta\alpha}(q - p) \\ &\quad \times \gamma^\beta L \nu(p), \end{aligned} \quad (21)$$

where g is the electroweak $SU(2)$ coupling constant of the standard model. Comparing this result with Eq. (4) one can express the neutrino self-energy operator as

$$\Sigma(p) = -\frac{ig^2}{2} \gamma^\alpha L J_{\alpha\beta}(p) \gamma^\beta L, \quad (22)$$

where

$$J_{\alpha\beta}(p) = \int \frac{d^4q}{(2\pi)^4} S(q) G_{\beta\alpha}(q - p). \quad (23)$$

To calculate $J_{\alpha\beta}(p)$ we need to unravel the propagators.

III. CHARGED-LEPTON AND W -BOSON PROPAGATOR IN A MAGNETIC FIELD

For the Fourier transform $S(q)$ of the translationally invariant part of the lepton propagator Eq. (18) one obtains in the Fock proper-time formalism [33]

$$S(q) = \int_0^\infty \frac{ds}{\cos(\beta s)} \exp \left[-is \left(m_\ell^2 - q_\parallel^2 + \frac{\tan(\beta s)}{\beta s} q_\perp^2 \right) \right] \left\{ [(q\gamma)_\parallel + m_\ell] \left[\cos(\beta s) - \frac{(\gamma\varphi\gamma)}{2} \sin(\beta s) \right] - \frac{(q\gamma)_\perp}{\cos(\beta s)} \right\}, \quad (24)$$

where $\beta = eB$ and m_ℓ is the lepton mass. Similarly, the Fourier transform of the translationally invariant part of the W -boson propagator Eq. (19) can be written in Feynman gauge as [27]

$$G_{\rho\sigma}(q) = - \int_0^\infty \frac{ds}{\cos(\beta s)} \exp \left[-is \left(m_W^2 - q_\parallel^2 + \frac{\tan(\beta s)}{\beta s} q_\perp^2 \right) \right] \left[(\tilde{\varphi}\tilde{\varphi})_{\rho\sigma} - (\varphi\varphi)_{\rho\sigma} \cos(2\beta s) - \varphi_{\rho\sigma} \sin(2\beta s) \right]. \quad (25)$$

Manipulations with the exact expressions Eqs. (24) and (25) are extremely cumbersome. Magnetic fields existing in Nature probably are always weak compared with the critical field for the W -boson, $m_W^2/e \simeq 10^{24}$ G. Therefore, the W propagator can be expanded in powers of β as a small parameter. We find up to second order

$$G_{\rho\sigma}(q) = -i \frac{g_{\rho\sigma}}{q^2 - m_W^2} - \beta \frac{2\varphi_{\rho\sigma}}{(q^2 - m_W^2)^2} + i\beta^2 \left[g_{\rho\sigma} \left(\frac{1}{(q^2 - m_W^2)^3} + \frac{2q_\perp^2}{(q^2 - m_W^2)^4} \right) + 4(\varphi\varphi)_{\rho\sigma} \frac{1}{(q^2 - m_W^2)^3} \right] + \mathcal{O}(\beta^3). \quad (26)$$

Likewise, the asymptotic expression for the lepton propagator $S(q)$ is realised when the field strength is the smallest dimensional parameter, $\beta \ll m_\ell^2 \ll m_W^2$. In this “weak field approximation” the charged-lepton propagator can be expanded as [34]

$$S(q) = i \frac{(q\gamma) + m_\ell}{q^2 - m_\ell^2} + \beta \frac{(q\gamma)_\parallel + m_\ell}{2(q^2 - m_\ell^2)^2} (\gamma\varphi\gamma) + \beta^2 \frac{2i \left[(q_\parallel^2 - m_\ell^2) (q\gamma)_\perp - q_\perp^2 ((q\gamma)_\parallel + m_\ell) \right]}{(q^2 - m_\ell^2)^4} + \mathcal{O}(\beta^3). \quad (27)$$

One can see from this expansion that the contribution of the region of small virtual momenta $q^2 \sim m_\ell^2 \ll m_W^2$ is enhanced in each succeeding term. If the propagator is used for a “moderate field,” $m_\ell^2 \ll \beta \ll m_W^2$, the expansion is not applicable and the exact propagator Eq. (24) must be used.

When the magnetic field is strong enough, $B \gtrsim B_\ell = m_\ell^2/e$, another possibility is to express the charged-lepton propagator as an expansion in terms of Landau levels [35]

$$S(q) = \sum_{n=0}^{\infty} \frac{i}{q_\parallel^2 - m_\ell^2 - 2n\beta} \left\{ [(q\gamma)_\parallel + m_\ell] \left[d_n(v) - \frac{i}{2} (\gamma\varphi\gamma) d'_n(v) \right] - (q\gamma)_\perp 2n \frac{d_n(v)}{v} \right\}, \quad (28)$$

where $v = q_\perp^2/\beta$ and

$$d_n(v) = (-1)^n e^{-v} [L_n(2v) - L_{n-1}(2v)]. \quad (29)$$

Here, $L_n(x)$ are the Laguerre polynomials with the additional definition $L_{-1}(x) = 0$.

IV. CONTRIBUTION OF THE LEPTON LOW LANDAU LEVELS

As we have already stressed in the introduction, our final result derived in Sec V below strongly disagrees with that of Refs. [20, 21]. We think that the disagreement arises because these authors use only one lowest Landau level in the charged-lepton propagator in the case of moderate field strengths which they call “strong fields.” However, the contributions of the next Landau levels can be of the same order as the ground-level contribution because in the integration over the virtual lepton four-momentum in the loop the region $q_\parallel^2 \sim m_W^2 \gg \beta$ appears to be essential.

To substantiate this point we calculate the contribution to the neutrino self-energy operator from the n -th charged-lepton Landau level in conjunction with the exact W -propagator in the limit $p_\perp^2/m_W^2 \ll m_W^2/\beta$. Substituting the exact W -propagator Eq. (25) and the n -th charged-lepton Landau level from Eq. (28) into Eq. (23) we find

$$\begin{aligned} J_{\sigma\rho}^{(n)}(p) &= - \int \frac{d^4 q}{(2\pi)^4} \frac{i}{q_\parallel^2 - m_\ell^2 - 2n\beta} \left\{ (q\gamma)_\parallel \left[d_n(v) - \frac{i}{2} (\gamma\varphi\gamma) d'_n(v) \right] - (q\gamma)_\perp 2n \frac{d_n(v)}{v} \right\} \\ &\times \int_0^\infty \frac{ds}{\cos(\beta s)} \exp \left[-is \left(m_W^2 - (q-p)_\parallel^2 + \frac{\tan(\beta s)}{\beta s} (q-p)_\perp^2 \right) \right] \left[(\tilde{\varphi}\tilde{\varphi})_{\rho\sigma} - (\varphi\varphi)_{\rho\sigma} \cos(2\beta s) - \varphi_{\rho\sigma} \sin(2\beta s) \right]. \end{aligned} \quad (30)$$

Terms with even numbers of γ matrices were omitted because they are removed by the chiral structure of the operator Eq. (22). Next we perform a clockwise rotation in the complex plane $s = -i\tau$ and use the identity

$$\frac{1}{q_\parallel^2 - m_\ell^2 - 2n\beta} = - \int_0^\infty d\tau' \exp \left[-\tau' (m_\ell^2 + 2n\beta - q_\parallel^2) \right]. \quad (31)$$

These manipulations allow us to rewrite the integral Eq. (30) as

$$\begin{aligned}
J_{\sigma\rho}^{(n)}(p) &= \int \frac{d^4 q}{(2\pi)^4} \left\{ (q\gamma)_{\parallel} \left[d_n(v) - \frac{i}{2} (\gamma\varphi\gamma) d'_n(v) \right] - (q\gamma)_{\perp} 2n \frac{d_n(v)}{v} \right\} \\
&\times \int_0^\infty \frac{d\tau d\tau'}{\cosh(\beta\tau)} \left[(\tilde{\varphi}\tilde{\varphi})_{\rho\sigma} - (\varphi\varphi)_{\rho\sigma} \cosh(2\beta\tau) + i \varphi_{\rho\sigma} \sinh(2\beta\tau) \right] \\
&\times \exp \left[-\tau' (m_\ell^2 + 2n\beta - q_{\parallel}^2) - \tau (m_W^2 - (q-p)_{\parallel}^2) - \frac{\tanh(\beta\tau)}{\beta} (q-p)_{\perp}^2 \right]. \quad (32)
\end{aligned}$$

In the integration over $d^4 q = d^2 q_{\parallel} d^2 q_{\perp}$, the integrals over $d^2 q_{\parallel}$ can be easily calculated because they are of Gaussian form. As a result we find

$$\begin{aligned}
J_{\sigma\rho}^{(n)}(p) &= \frac{i}{16\pi^3 m_W^2} \int_0^\infty \frac{dx dy}{(x+y) \cosh(\eta x)} \exp \left[-x + \xi \frac{xy}{x+y} - y(2n\eta + \lambda) \right] \\
&\times \left[(\tilde{\varphi}\tilde{\varphi})_{\rho\sigma} - (\varphi\varphi)_{\rho\sigma} \cosh(2\eta x) + i \varphi_{\rho\sigma} \sinh(2\eta x) \right] \\
&\times \int d^2 q_{\perp} \exp \left[-\frac{\tanh(\eta x)}{\beta} (q-p)_{\perp}^2 \right] \left\{ (p\gamma)_{\parallel} \frac{x}{x+y} \left[d_n(v) - \frac{i}{2} (\gamma\varphi\gamma) d'_n(v) \right] - (q\gamma)_{\perp} 2n \frac{d_n(v)}{v} \right\}, \quad (33)
\end{aligned}$$

where the dimensionless variables $x = m_W^2 \tau$ and $y = m_W^2 \tau'$ have been introduced as well as the parameters $\eta = \beta/m_W^2$, $\xi = p_{\parallel}^2/m_W^2 \simeq p_{\perp}^2/m_W^2$ and $\lambda = m_\ell^2/m_W^2$. From Eq. (33) follows that the essential region of the x variable is $x \sim 1$ due to the exponential e^{-x} . Given the condition $\eta \ll 1$, the argument of the hyperbolic functions is small, $\eta x \ll 1$, leading to an obvious simplification. One should also take into account the condition $q_{\perp}^2 \sim \beta$ caused by the functions $d_n(v)$, see Eq. (29), containing the exponential e^{-v} . For a wide range of the numbers n the exponential in the integral over $d^2 q_{\perp}$ is simplified, with the only restriction $n \ll 1/\eta = m_W^2/\beta$:

$$\exp \left[-\frac{\tanh(\eta x)}{\beta} (q-p)_{\perp}^2 \right] \simeq \exp \left(-x \frac{p_{\perp}^2}{m_W^2} \right) \times \exp \left(-x \frac{q_{\perp}^2 - 2(qp)_{\perp}}{m_W^2} \right). \quad (34)$$

Here, the first exponential is equal to $e^{-\xi x}$. We consider the value p_{\perp}^2 to vary in a very wide range, $0 < p_{\perp}^2 \ll m_W^4/\beta$. The second exponential is equal to unity with a good accuracy, because $q_{\perp}^2 \sim \beta \ll m_W^2$ and $(qp)_{\perp} \ll m_W^2$. With these approximations, the integration over $d^2 q_{\perp}$ can be easily performed,

$$\int d^2 q_{\perp} d_n(v) = \pi \beta (2 - \delta_{n0}), \quad \int d^2 q_{\perp} d'_n(v) = -\pi \beta \delta_{n0}, \quad \int d^2 q_{\perp} (q\gamma)_{\perp} \frac{d_n(v)}{v} = 0. \quad (35)$$

The investigated integral acquires the form

$$J_{\sigma\rho}^{(n)}(p) = \frac{i\beta}{16\pi^2 m_W^2} (p\gamma)_{\parallel} g_{\rho\sigma} \left\{ 2 - \left[1 - \frac{i}{2} (\gamma\varphi\gamma) \right] \delta_{n0} \right\} \int_0^\infty \frac{x dx dy}{(x+y)^2} \exp \left[-x - \xi \frac{x^2}{x+y} - y(2n\eta + \lambda) \right]. \quad (36)$$

Taking into account the smallness of the parameters η and λ , one finally obtains for $n \ll m_W^2/\beta$

$$J_{\sigma\rho}^{(n)}(p) = \frac{i\beta}{16\pi^2 p_{\perp}^2} \ln \left(1 + \frac{p_{\perp}^2}{m_W^2} \right) (p\gamma)_{\parallel} g_{\rho\sigma} \left\{ 2 - \left[1 - \frac{i}{2} (\gamma\varphi\gamma) \right] \delta_{n0} \right\}. \quad (37)$$

Substituting Eq. (37) into Eq. (22) we finally find the contribution of the n -th Landau level of the lepton propagator to the neutrino self-energy operator

$$\Sigma^{(n)}(p) = -\frac{G_F e B}{\sqrt{2} 2\pi^2} \frac{m_W^2}{p_{\perp}^2} \ln \left(1 + \frac{p_{\perp}^2}{m_W^2} \right) [(2 - \delta_{n0}) (p\gamma)_{\parallel} - \delta_{n0} (p\tilde{\varphi}\gamma)] L. \quad (38)$$

We conclude from Eq. (38) that, contrary to the treatment of Refs. [20, 21], the lowest Landau level does not dominate.

For higher Landau levels, $n \gtrsim m_W^2/\beta$, the calculation is more cumbersome. Therefore, using the lepton propagator expanded in terms of the Landau levels, with a further summation, is extremely inconvenient. It is much simpler to take the exact lepton propagator in the form of Eq. (24). This approach is used in Sec V below.

In Ref. [21] a numerical test of the lowest Landau level domination was made. However, we believe that the results of this test are misleading. In the right-hand side of Fig. 2 of Ref. [21], corresponding to strong fields $eB \simeq 10^4 m_e^2$, the value \varkappa_a defined in Eq. (89) of Ref. [21] tends to unity. From this behavior it was concluded that the lowest Landau level domination worked well in this region. However, the left-hand side of that plot, corresponding to weak fields $eB \simeq 10^{-4} m_e^2$, would have to coincide with, but strongly contradicts, the weak-field result of Ref. [27]. An accurate analysis of Eqs. (89) and (90) of Ref. [21] shows that a precise cancellation of the two infinities arises, whereas the rest appears to be of order $eB/m_W^2 \lesssim 10^{-6}$ for the relevant field values, but not of order unity as claimed there.

V. NEW CALCULATION OF $\Sigma(p)$

We begin our calculation of $\Sigma(p)$ with the simpler case of a “weak field” where B defines the smallest energy scale of the problem, $eB \ll m_\ell^2 \ll m_W^2$. As an essential simplification we use the field expansions of the Fourier transformed W and lepton propagators. Substituting Eqs. (27) and (26) in Eq. (23), and assuming in addition that $p_\perp^2 \ll m_W^2$, we find an expansion of $J_{\alpha\beta}(p)$ in powers of the field strength,

$$J_{\alpha\beta}(p) = \frac{1}{16\pi^2} \left\{ \frac{eB}{m_W^2} \left[-\frac{i}{2} g_{\alpha\beta}(p\tilde{\varphi}\gamma)\gamma_5 + \varphi_{\alpha\beta}(p\gamma) \right] + i \left(\frac{eB}{m_W^2} \right)^2 \left[g_{\alpha\beta}(p\gamma)_\parallel \left(\frac{2}{3} \ln \frac{m_W^2}{m_\ell^2} - \frac{3}{2} \right) + i \varphi_{\alpha\beta}(p\tilde{\varphi}\gamma)\gamma_5 - (\varphi\varphi)_{\alpha\beta}(p\gamma) \right] \right\} + \dots \quad (39)$$

The dots include terms having the structure $g_{\alpha\beta}(p\gamma)$ and contain, in particular, the ultraviolet divergence, to be fully absorbed by the neutrino wave-function renormalization. They also include terms with an even number of γ matrices that are removed by the chiral structure of $\Sigma(p)$.

Substituting Eq. (39) into Eq. (22) we finally find

$$\Sigma(p) = \frac{G_F}{\sqrt{2}4\pi^2} \left[3e(p\tilde{F}\gamma) - \frac{e^2(p\tilde{F}\tilde{F}\gamma)}{m_W^2} \left(\frac{4}{3} \ln \frac{m_W^2}{m_\ell^2} + 1 \right) \right] L. \quad (40)$$

The corresponding coefficients b and c are shown in Table I.

For a “moderate field,” $m_\ell^2 \ll eB \ll m_W^2$ we use the exact expression Eq. (24) for the charged-lepton propagator and the expansion Eq. (26) for the W propagator, with the same assumption $p_\perp^2 \ll m_W^2$. After a straightforward but cumbersome calculation we find

$$J_{\alpha\beta}(p) = \frac{1}{16\pi^2} \left\{ \frac{eB}{m_W^2} \left[-\frac{i}{2} g_{\alpha\beta}(p\tilde{\varphi}\gamma)\gamma_5 + \varphi_{\alpha\beta}(p\gamma) \right] + i \left(\frac{eB}{m_W^2} \right)^2 \left[g_{\alpha\beta}(p\gamma)_\parallel \left(\frac{2}{3} \ln \frac{m_W^2}{eB} - \frac{7}{6} + \frac{1}{3} \ln 3 + \frac{2}{3} \gamma_E - 2I \right) + i \varphi_{\alpha\beta}(p\tilde{\varphi}\gamma)\gamma_5 - (\varphi\varphi)_{\alpha\beta}(p\gamma) \right] \right\} + \dots, \quad (41)$$

where $\gamma_E = 0.577\dots$ is the Euler constant, and

$$I = \int_0^\infty \frac{dz}{z^3} \left(\frac{z^2}{\sinh^2 z} - \frac{3}{3+z^2} \right) \approx -0.055. \quad (42)$$

The presence of the term $\beta^2 \ln \beta$ with $\beta = eB$ in Eq. (41) shows that an expansion of the lepton propagator in powers of β as a small parameter is not possible.

Finally we find

$$\Sigma(p) = \frac{G_F}{\sqrt{2}4\pi^2} \left[3e(p\tilde{F}\gamma) - \frac{e^2(p\tilde{F}\tilde{F}\gamma)}{m_W^2} \left(\frac{4}{3} \ln \frac{m_W^2}{eB} + 3.389 \right) \right] L. \quad (43)$$

Again, the corresponding coefficients b and c are shown in Table I.

VI. SUMMARY

We have calculated the neutrino self-energy operator $\Sigma(p)$ in a magnetic field at one-loop order. Our results for the invariant coefficients of Eq. (5) that characterize $\Sigma(p)$ are shown in Table I for weak and moderate field strengths and are compared to those of previous authors. Our results strongly disagree with those of the recent Refs. [20, 21] where a large effect was found. For

moderate fields we have shown that considering only the lowest Landau level contribution in the lepton propagator is incorrect because the contributions of the next Landau levels are of similar magnitude.

It is instructive to reproduce explicitly the energy of a ν_e or $\bar{\nu}_e$ in the presence of a CP symmetric plasma and the simultaneous presence of a magnetic field. We write the energy shift in the form of Eq. (2). For a weak field $eB \ll m_e^2$ we find

$$\frac{E}{|\mathbf{p}|} = 1 + \frac{\sqrt{2}G_F}{3} \left[-\frac{7\pi^2 T^4}{15} \left(\frac{1}{m_Z^2} + \frac{2}{m_W^2} \right) + \frac{T^2 eB}{m_W^2} \cos \phi + \frac{(eB)^2}{2\pi^2 m_W^2} \sin^2 \phi \left(\ln \frac{T^2}{m_e^2} - \ln \frac{m_W^2}{m_e^2} - \frac{3}{4} \right) \right]. \quad (44)$$

Interestingly, the logarithmic B -field induced plasma term in the third term and the logarithmic pure B -field term add to $\ln(T^2/m_W^2)$ so that no electron-mass dependence remains. For a moderate field $m_e^2 \ll eB \ll m_W^2$ the third term is

$$+ \frac{(eB)^2}{2\pi^2 m_W^2} \sin^2 \phi \left(\ln \frac{T^2}{m_e^2} - \ln \frac{m_W^2}{eB} - 2.542 \right). \quad (45)$$

In this case an electron-mass dependence remains.

In a plasma, the pure B -field term is comparable to the logarithmic contribution of the B -field induced plasma term derived in Ref. [17]. However, these logarithmic terms do not seem to be numerically important relative to the term linear in eB .

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